Circularly Symmetric Optical Waveguide with Strong Anisotropy

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***Abstract*—A solution to the problem of wave propagation in an anisotropic, circularly symmetric optical waveguide is presented. Exact solutions are given for the step-index case when the core and the cladding consist of uniaxial materials with their optical axes parallel with the axis of the cylindrical waveguide.**

I. Introduction

With the considerable efforts that are currently being made of making single crystal optical fibers, there is a need of knowing the mode structure of dielectric guides consisting of anisotropic materials. However, finding the modes in an anisotropic medium with a cylindrical boundary is, in general, a complicated problem. This is so because the translational symmetry of a crystal does not naturally lend itself to a simple description in terms of the cylindrical coordinates required by the boundary conditions. Approximate solutions have been given for cases of weak anisotropy caused by elastic deformations or thermoelastic stresses of a medium which is isotropic in its undeformed state [1]. The purpose here is to discuss and solve the simplest case of strong anisotropy appearing in a guide consisting of monocrystalline media. The simplest case is, no doubt, that of a uniaxial medium with the optical axis coinciding with the cylinder axis. It will be shown that in this case a step-index circular waveguide having anisotropic materials for core and cladding has modes that are simple generalizations of those for the isotropic waveguide. As a numerical example, solutions for the lowest order modes of a lithium niobate waveguide are given.

II. Formulation of the Problem

When the optical axis of a uniaxial material is taken to be the z axis, the nonzero components of the dielectric tensor are

$$\begin{aligned}\epsilon_{xx} &= \epsilon_0 n^2\\ \epsilon_{yy} &= \epsilon_0 n^2\\ \epsilon_{zz} &= \epsilon_0 f^2 n^2. \end{aligned} \tag{1}$$

Here, f is a factor determining the degree of anisotropy, the isotropic case corresponding to $f=1$.

Consider a cylindrical waveguide with a stepwise change of refractive index at radius

$$r = a_0. \tag{2}$$

The z axis is the axis of the cylinder. Suppose that the core of the waveguide, the region defined by

$$r \leqslant a_0$$

has optical properties as given by (1). Outside this region is the cladding with optical properties given by

$$\left.\begin{aligned}\epsilon_{xx} &= \epsilon_0 n_2^2\\ \epsilon_{yy} &= \epsilon_0 n_2^2\\ \epsilon_{zz} &= \epsilon_0 g^2 n_2^2\end{aligned}\right\} \quad (r > a_0). \tag{3}$$

Introducing now the cylindrical coordinate system

$$(r, \theta, z)$$

we may write the relation between the components of $\boldsymbol{D}$ and $\boldsymbol{E}$ in the cladding as

$$\begin{pmatrix} D_r \\ D_\theta \end{pmatrix} = n_2^2 \epsilon_0 \begin{pmatrix} E_r \\ E_\theta \end{pmatrix}$$

$$D_z = g^2 n_2^2 \epsilon_0 E_z. \tag{4}$$

Maxwell's equations for monochromatic fields of angular frequency ω are

$$\begin{aligned}\nabla \times \boldsymbol{E} + j\omega\mu_0 \boldsymbol{H} &= 0\\ \nabla \times \boldsymbol{H} - j\omega \boldsymbol{D} &= 0. \end{aligned} \tag{5}$$

We attempt to find solutions for the fields in the form

$$\boldsymbol{F}(r) e^{j(\nu\theta - \beta z)} \tag{6}$$

where

$$\nu = 0, \pm 1, \pm 2, \cdots$$

and $\boldsymbol{F}$ represents any of the field vectors $\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{H}$. β is the constant of propagation. When the curls are expressed in terms of cylindrical coordinates, (5) is seen to be equivalent to six scalar equations, of which four are differential equations of first order, while the remaining two are algebraic equations. The latter may be used to eliminate E_r and H_r from the differential equations. We are then left with four coupled first-order equations in the variables E_θ, E_z, H_θ, and H_z.

Manuscript received July 14, 1981; revised September 29, 1981. This work was supported by the Royal Norwegian Council for Scientific and Industrial Research.

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For the isotropic case it has been shown [2] to be advantageous to introduce as vector variable

$$w(s)=\begin{pmatrix} sE_\theta/Z_0^{1/2} \\ E_z/Z_0^{1/2} \\ sH_\theta Z_0^{1/2} \\ H_z Z_0^{1/2} \end{pmatrix} \tag{7}$$

where

$$Z_0=(\mu_0/\epsilon_0)^{1/2} \tag{8}$$

s is the normalized radius

$$s=rk_0=r\frac{\omega}{c} \tag{9}$$

and c is the velocity of light in vacuum.

The set of coupled differential equations may then be written

$$\frac{d}{ds}w(s)-Mw(s)=0 \tag{10}$$

where the system matrix M is given by

$$M=\frac{j}{s}\begin{pmatrix} 0 & 0 & \nu b/n_2^2 & (\nu^2-n_2^2s^2)/n_2^2 \\ 0 & 0 & (n_2^2-b^2)/n_2^2 & -\nu b/n_2^2 \\ -\nu b & -(\nu^2-g^2n_2^2s^2) & 0 & 0 \\ -(n_2^2-b^2) & \nu b & 0 & 0 \end{pmatrix}. \tag{11}$$

Here, b is the normalized constant of propagation

$$b=\beta/k_0.$$

From the two algebraic equations resulting from (5), the radial components may be expressed in terms of $w(s)$

$$\begin{pmatrix} E_r/Z_0^{1/2} \\ H_r Z_0^{1/2} \end{pmatrix}=\begin{pmatrix} 0 & 0 & b/n_2^2 & \nu/sn_2^2 \\ -b & -\nu/s & 0 & 0 \end{pmatrix}w(s). \tag{12}$$

The system matrix M has here, as in the isotropic case [2], the symmetry property

$$\sigma M\sigma=-\tilde{M}^* \tag{13}$$

where

$$\sigma=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}. \tag{14}$$

The dots stand for 0, $\tilde{M}$ is the transpose of M, and the star denotes the complex conjugate. As shown in detail in [2], this property of M leads to the result that for any two solutions of (10), say $w_1(s)$ and $w_2(s)$, we have

$$\tilde{w}_1^*(s)\sigma w_2(s)=\text{const.} \tag{15}$$

To obtain the equations corresponding to (11) and (12) for the core region we replace n_2 and g by n and f, respectively.

III. Solution of the Differential Equation

The equation (10) always has four independent vector solutions. Although the equation is valid when n_2 and g are arbitrary functions of s, we shall here confine discussion to the case of n_2 and g being constant, and show that solutions of (1) may then be obtained from those of the isotropic waveguide.

The system matrix for the isotropic case is obtained from (11) when the factor of anisotropy g is given the value 1. Let us denote the system matrix obtained in this way by

$$M_I=M_{g=1}. \tag{16}$$

The differential equation for the isotropic case then is

$$\frac{d}{ds}v-M_Iv=0. \tag{17}$$

Its solutions are well known and were first derived by Snitzer [3].

Let us first assume

$$b>n_2 \tag{18}$$

and introduce

$$\rho=(b^2-n_2^2)^{1/2} \tag{19}$$

which is then a real quantity. There are two independent solutions of (17) that are regular at $s=\infty$. These are [2]

$$v_1(s)=\begin{bmatrix} -j\dfrac{\nu b}{n_2\rho}K_\nu(\rho s) \\ j\dfrac{\rho}{n_2}K_\nu(\rho s) \\ -n_2sK_\nu'(\rho s) \\ 0 \end{bmatrix} \qquad v_2(s)=\begin{bmatrix} jsK_\nu'(\rho s) \\ 0 \\ \dfrac{\nu b}{\rho}K_\nu(\rho s) \\ -\rho K_\nu(\rho s) \end{bmatrix}. \tag{20}$$

Here, K_ν is the modified Bessel function of second kind and order ν. We note that the two solutions (2) are E-waves and H-waves, respectively.

To find the solutions for the anisotropic case we first note that since $v_2(s)$ satisfies (17) we have

$$\frac{d}{ds}v_2(s)=M_I(s)v_2(s). \tag{21}$$

Only one of M_I's elements is different from the corresponding element of M, namely, the element in line 3 and column 2. However, the right-hand side of (21) is seen to be independent of this element, because the second of v_2's components is zero. Therefore, (21) remains valid if we exchange $\boldsymbol{M}_I$ for $\boldsymbol{M}$

$$\frac{d}{ds}\boldsymbol{v}_2(s)=\boldsymbol{M}(s)\boldsymbol{v}_2(s). \tag{22}$$

This shows that $\boldsymbol{v}_2(s)$ is a solution of (10). The H-wave, having no axial electric field component, is obviously not affected by the type of anisotropy discussed here.

Turning now to the function $\boldsymbol{v}_1(s)$, we know that it satisfies the differential equation

$$\frac{d}{d(gs)}\boldsymbol{v}_1(gs)=\boldsymbol{M}_I(gs)\boldsymbol{v}_1(gs). \tag{23}$$

This slightly modified version of (17) is obtained by change of the dependent variable from s to gs. Since g is a constant, this amounts to a change of scale in the independent variable. The latter equation may also be written

$$\frac{d}{ds}\boldsymbol{v}_1(gs)=g\boldsymbol{M}_I(gs)\boldsymbol{v}_1(gs). \tag{24}$$

It is easily seen that the elements of

$$g\boldsymbol{M}_I(gs)$$

equal those of $\boldsymbol{M}(s)$ with one exception: the element of line 1 and column 4. However, the right-hand side of (24) does not depend on this element, due to the fact that the fourth component of $\boldsymbol{v}_1(s)$ is zero. Equation (24), therefore, remains valid if the system matrix

$$g\boldsymbol{M}_I(gs)$$

is replaced by $\boldsymbol{M}(s)$. We then have

$$\frac{d}{ds}\boldsymbol{v}_1(gs)=\boldsymbol{M}(s)\boldsymbol{v}_1(gs) \tag{25}$$

which shows that $\boldsymbol{v}_1(gs)$ is a solution of (10).

To sum up, we have shown that two solutions of (10), regular at $s=\infty$, are

$$\boldsymbol{u}_1(s)=\begin{bmatrix} -j\dfrac{\nu b}{n_2\rho}K_\nu(\rho sg) \\ j\dfrac{\rho}{n_2}K_\nu(\rho gs) \\ -n_2 gsK_\nu'(\rho gs) \\ 0 \end{bmatrix} \qquad \boldsymbol{u}_2(s)=\begin{bmatrix} jsK_\nu'(\rho s) \\ 0 \\ \dfrac{\nu b}{\rho}K_\nu(\rho s) \\ -\rho K_\nu(\rho s) \end{bmatrix} \tag{26}$$

with ρ given by (19). The above solutions are relevant for the cladding of a step-index anisotropic waveguide since they are regular at infinity.

Two other solutions are found from (26) when K_ν is replaced by I_ν, the modified Bessel function of first kind and order ν

$$\boldsymbol{u}_3(s)=\begin{bmatrix} \dfrac{\nu b}{n_2\rho}I_\nu(\rho gs) \\ -\dfrac{\rho}{n_2}I_\nu(\rho gs) \\ -jn_2 gsI_\nu'(\rho gs) \\ 0 \end{bmatrix} \qquad \boldsymbol{u}_4(s)=\begin{bmatrix} -sI_\nu'(\rho s) \\ 0 \\ j\dfrac{\nu b}{\rho}I_\nu(\rho s) \\ -j\rho I_\nu(\rho s) \end{bmatrix}. \tag{27}$$

The functions (26) and (27) together form a complete set of solutions of (10) and satisfy the orthogonality conditions

$$\tilde{\boldsymbol{u}}_i^*\boldsymbol{\sigma}\boldsymbol{u}_j=\begin{cases}1, & \text{if } (i,j)=(1,3) \text{ or } (2.4) \\ 0, & \text{otherwise.}\end{cases} \tag{28}$$

These relations are a consequence of the symmetry property (13). To see that (28) is correct, we note on one hand that, according to (15), each of these products is a constant. Its value is most easily found by evaluation of the limit of the products for $s=0$ or $s=\infty$.

The solutions for the core region may now be obtained from the ones discussed above, exchanging n_2 and g for n and f. We also introduce

$$\gamma=(n^2-b^2)^{1/2} \tag{29}$$

assuming γ to be real. Comparing this with (19), we observe that ρ corresponds to $j\gamma$. In order to transform (27) into a set of solutions valid for the core region we must accordingly replace ρ by $j\gamma$.

The modified Bessel functions with imaginary argument may now be replaced by ordinary Bessel functions with real argument [4]. The relation between these functions may be written

$$I_\nu(jz)=(-1)^{\nu+1}j^\nu J_\nu(z). \tag{30}$$

For a material with index n and anisotropy factor f (instead of n_2 and g) we obtain from (27) the two solutions

$$\boldsymbol{w}_1(s)=\begin{bmatrix} \nu bJ_\nu(\gamma fs) \\ \gamma^2 J_\nu(\gamma fs) \\ -j\gamma fsn^2J_\nu'(\gamma fs) \\ 0 \end{bmatrix} \qquad \boldsymbol{w}_2(s)=\begin{bmatrix} -\gamma sJ_\nu'(\gamma s) \\ 0 \\ j\nu bJ_\nu(\gamma s) \\ j\gamma^2J_\nu(\gamma s) \end{bmatrix}. \tag{31}$$

$\boldsymbol{w}_1(s)$ and $\boldsymbol{w}_2(s)$ are regular at $s=0$ and any vector representing the fields in the core region must be a linear combination of these two functions.

IV. The Guided Modes

Let us introduce the normalized core radius

$$a=k_0a_0=\frac{\omega}{c}a_0. \tag{32}$$

The field in the core may be written

$$\boldsymbol{w}(s)=\alpha_E\boldsymbol{w}_1(s)+\alpha_H\boldsymbol{w}_2(s), \qquad s\lessgtr a. \tag{33}$$

In the cladding, any solution of (10) may be written as a linear combination of $\boldsymbol{u}_1(s),\ldots,\boldsymbol{u}_4(s)$. However, the fields must tend to zero at infinity and, therefore, the coefficients of expansion of $\boldsymbol{u}_3(s)$ and $\boldsymbol{u}_4(s)$ must be zero. This condition together with the boundary condition at $s=0$ and the orthogonality relations (28) lead to

$$\begin{pmatrix} \tilde{\boldsymbol{u}}_1^*(a)\boldsymbol{\sigma}\boldsymbol{w}_1(a) & \tilde{\boldsymbol{u}}_1^*(a)\boldsymbol{\sigma}\boldsymbol{w}_2(a) \\ \tilde{\boldsymbol{u}}_2^*(a)\boldsymbol{\sigma}\boldsymbol{w}_1(a) & \tilde{\boldsymbol{u}}_2^*(a)\boldsymbol{\sigma}\boldsymbol{w}_2(a) \end{pmatrix}\begin{pmatrix}\alpha_E \\ \alpha_H\end{pmatrix}=0. \tag{34}$$

Requiring now the determinant of (34) to be zero, we obtain the dispersion relation which determines the normalized constant of propagation b. This form of the relation is well suited for computational purposes. For the purpose of carrying out an analytical discussion we insert for the functions $\boldsymbol{u}$ and $\boldsymbol{w}$ from (26) and (31) and obtain

$$\left[\epsilon f^2 \frac{1}{\gamma fa}\frac{J_\nu'(\gamma fa)}{J_\nu(\gamma fa)} + g^2 \frac{1}{\rho ga}\frac{K_\nu'(\rho ga)}{K_\nu(\rho ga)}\right] \cdot \left[\frac{1}{\gamma a}\frac{J_\nu'(\gamma a)}{J_\nu(\gamma a)} + \frac{1}{\rho a}\frac{K_\nu'(\rho a)}{K_\nu(\rho a)}\right] = \left[\frac{\nu b}{\gamma^2\rho^2}\frac{n_2}{a^2}(\epsilon-1)\right]^2. \tag{35}$$

Here we have put

$$\epsilon = \frac{n^2}{n_2^2}. \tag{36}$$

This is obviously a generalization of the well-known relation for isotropic step-index waveguides. Schlesinger, Diament, and Vigants [5] have shown how to recast the equation to a form more suitable for discussing the cutoff frequencies of the modes. Following the example of these authors and using, as far as possible., their notation, we introduce

$$J^+(z) = \frac{1}{z}\frac{J_{\nu+1}(z)}{J_\nu(z)} \qquad J^-(z) = \frac{1}{z}\frac{J_{\nu-1}(z)}{J_\nu(z)}$$
$$K^+(z) = \frac{1}{z}\frac{K_{\nu+1}(z)}{K_\nu(z)} \qquad K^-(z) = \frac{1}{z}\frac{K_{\nu-1}(z)}{K_\nu(z)}. \tag{37}$$

When the derivatives in (35) are eliminated by means of

$$J_\nu' = \tfrac{1}{2}(J_{\nu-1} - J_{\nu+1})$$

and

$$K_\nu' = -\tfrac{1}{2}(K_{\nu-1} + K_{\nu+1}) \tag{38}$$

(see [4]), the left-hand side of (35) may obviously be expressed in terms of the functions defined above. In [5] it is shown that when use is made of the recursion formulas for Bessel functions this may be done also for the right-hand side. Following a similar procedure, we find for the dispersion relation

$$\left[\epsilon f^2 J^-(\gamma fa) - g^2 K^-(\rho ga)\right]\left[J^+(\gamma a) + K^+(\rho a)\right] + \left[\epsilon f^2 J^+(\gamma fa) + g^2 K^+(\rho ga)\right]\left[J^-(\gamma a) - K^-(\rho a)\right] = 0. \tag{39}$$

A. Cutoff Conditions

The above equation is well suited for analyzing the cutoff conditions for the various modes. At cutoff, ρ will tend to zero

$$\rho \to 0.$$

It follows from (19) that

$$b \to n_2 \tag{40}$$

and from (29) that

$$\gamma \to \gamma_c = \left(n^2 - n_2^2\right)^{1/2}. \tag{41}$$

From the properties of Bessel functions for small arguments [4] it follows that when $z \to 0$ we have

$$K^+(z) \to \begin{cases} -\dfrac{1}{z^2 \ln z/2}, & \text{for } \nu = 0 \\ \dfrac{2\nu}{z^2}, & \text{for } \nu > 0 \end{cases}$$

$$K^-(z) \to \begin{cases} -\dfrac{1}{z^2 \ln z/2}, & \text{for } \nu = 0 \\ -\ln z/2, & \text{for } \nu = 1 \\ \dfrac{1}{2(\nu-1)}, & \text{for } \nu > 1. \end{cases} \tag{42}$$

Let us now consider the forms taken by (39) for various values of ν.

Case 1. $\nu = 0$: From elementary properties of Bessel functions it follows that in this case

$$J^-(z) = -J^+(z)$$
$$K^-(z) = K^+(z). \tag{43}$$

This simplifies the dispersion relation (39) to

$$\left[\epsilon f^2 J^+(\gamma fa) + g^2 K^+(\rho ga)\right]\left[J^+(\gamma a) + K^+(\rho a)\right] = 0. \tag{44}$$

When $\rho \to 0$ it is seen from (42) that $K^+(\rho ga)$ and $K^+(\rho a)$ both tend to infinity. In order to satisfy (44) we must have either

$$J_0(\gamma_c a) = 0 \qquad (\mathrm{H}_{0m}\text{-modes}) \tag{45}$$

which determines the cutoff radii (and cutoff frequencies) of the H_{0m}-modes, or, we must have

$$J_0(\gamma_c fa) = 0 \qquad (\mathrm{E}_{0m}\text{-modes}). \tag{46}$$

Case 2. $\nu = 1$: When use is made of (42) it is seen that for $\rho \to 0$ the dominant term in (39) is

$$2/(\rho a)^2.$$

In order that the equation be satisfied it is necessary that the coefficient of this term is zero, i.e.

$$g^2 J^-(\gamma a) + g^2 \ln \frac{\rho a}{2} + \epsilon f^2 J^-(\gamma fa) + g^2 \ln \frac{\rho ga}{2} = 0. \tag{47}$$

This leads to

$$J_1(\gamma_c a) = 0 \qquad (\mathrm{HE}_{1m}\text{-modes}) \tag{48}$$

$$J_1(\gamma_c fa) = 0, \quad a \neq 0 \qquad (\mathrm{EH}_{1m}\text{-modes}). \tag{49}$$

Case 3. $\nu > 1$: Now the dominant term in (39) is

$$2\nu/(\rho a)^2.$$

Requiring that its coefficient be zero, we obtain

$$\epsilon f^2 J^-(\gamma_c fa) + J^-(\gamma_c a) = \frac{1+g^2}{2(\nu-1)}. \tag{50}$$

When these results are compared with those for an isotropic circular waveguide [5] and [6], we note that for the lowest order modes the cutoff frequencies of the E_{0m}

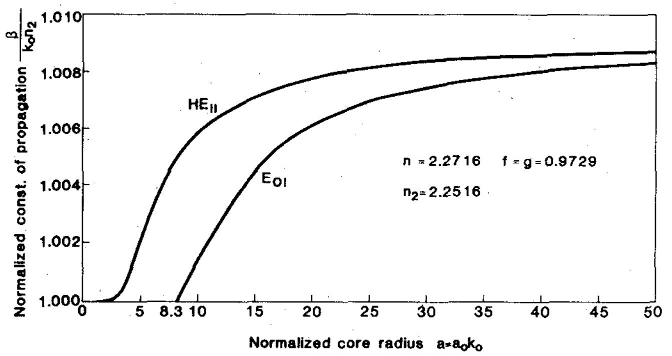

Fig. 1. Constant of propagation for the HE_{11}- and E_{01}-modes of an anisotropic waveguide as a function of core radius. The H_{01}-mode has cutoff radius 7.99 and its curve is too close to that of the E_{01}-mode to be plotted.

and EH_{1m}-modes are changed by anisotropy whereas the cutoff frequencies of the H_{0m} and HE_{1m} are not influenced. For the modes of higher order, the equation determining the cutoff frequencies (50), is considerably more complicated than in the isotropic case and involves the anisotropy factors of core and cladding.

In Fig. 1 is shown as an example the constant of propagation for two modes in a guide with a core of lithium-niobate.

V. Comments and Conclusion

The above discussion has been confined to what appears to be a very special case of an anisotropic waveguide. Only the step-index case has been treated and the assumption that the optical axis of both materials are parallel with the cylinder axis is evidently rather restrictive.

If we keep the assumption that the dielectric tensor has the form shown in (1), it is not difficult to extend the results obtained here to cover also the graded-index case, i.e., to let n and f in (1) be arbitrary functions of the distance from the axis. Yeh and Lindgren [7] have shown for the isotropic waveguide how the continuous variation of the index may be approximated by one which is piecewise constant. This allows the use of well-known solutions for the step-index case and appears to lead to efficient computational procedures. It is certainly possible to adapt this method to the anisotropic waveguide of the type discussed here and to the formalism used in this discussion.

If the simple anisotropy shown in (1) is replaced by a general anisotropy we meet the difficulty that the dielectric tensor, when converted to cylindrical coordinates, will have components that are periodic functions of the angle θ. If we, however, take the average value of the tensor with respect to the angle, we obtain a tensor of the form shown in (1) which may, accordingly, also be regarded as the first approximation to a more general type of anisotropy.

Acknowledgment

The work reported in this paper was done while the author was Visiting Professor at the University of California, Santa Barbara. The support and hospitality of the UCSB is gratefully acknowledged.

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